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A Bang-Bang Theorem for Optimization over Spaces of Analytic Functions

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Let $D = \{z: |z| \leqslant 1\}$ be the unit disk in the complex plane and let Π be the unit circle, the boundary of D. For a positive integer N, let $H^\infty(N)$ denote the space of boundary values of N-tuples of bounded holomorphic functions on D. Let $\Gamma(e^{i\theta}, w)$ be a function on $\Pi \times \mathbb{C}^N$, and consider the optimization problem:

Find

$$\gamma_0 = \inf_{h \in H^\infty(N)} \sup_{0 \leqslant \theta \leqslant 2\pi} \Gamma(e^{i\theta}, h(e^{i\theta})) = \inf_h \|\Gamma(\cdot, h)\|_\infty. \qquad \text{(OPT)}$$

Here $\| \ \|_\infty$ denotes the usual supremum norm for functions on Π. This article considers qualitative properties of an optimizing function h_0 for (OPT). In particular we give conditions on Γ which guarantee that $\Gamma(e^{i\theta}, h_0(e^{i\theta})) = \gamma_0$ for almost all θ. We call an (OPT) with this property *self flattening*. This problem has strong engineering motivation (see [H5]) which is illustrated in Section 4.

A classical example of (OPT) concerns $\Gamma(e^{i\theta}, w) = |f(e^{i\theta}) - w|^2$ for which it is well known that f continuous implies that $|f(e^{i\theta}) - h_0(e^{i\theta})|$ is constant a.e. in θ, see [G] for reference. Our results amount to generalizations of this result. In fact in Section 3 we use this classical linear result to prove nonlinear generalizations for differentiable Γ.

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COROLLARY (Sect. 3). *If h_0 in H^∞, an optimum for* (OPT), *is a continuous function* (*or is even near to one*) *and if* $\nabla_w \Gamma(e^{i\theta}, h_0(e^{i\theta})) \neq 0$ *for any* θ, *then* $\Gamma(e^{i\theta}, h_0(e^{i\theta})) = \gamma_0$.

This result says roughly that (unless degenerate) the optimum h_0 is pathological or it makes the objective function flat. This leads us to wonder what a priori conditions on Γ automatically guarantee self flattening. We give some results on this in terms of the sublevel sets

$$S(\theta, t) = \{w \in \mathbb{C}^N : \Gamma(e^{i\theta}, w) \leqslant t\}, \qquad \theta, t \in \mathbb{R}. \tag{0.1}$$

We shall say Γ is *large at* ∞ if $\bigcup_\theta S(\theta, t)$ is bounded for each t. An easy normal families argument shows that if Γ is large at ∞, there is an $h_0 \in H^\infty(N)$ which solves (OPT) in the sense that $\|\Gamma(\cdot, h_0(\cdot))\|_\infty = \gamma_0$.

Our main conclusion (Theorem 4) is that if each $S(\theta, \gamma_0)$ can be contracted to a point with a well behaved family of holomorphic maps F_θ which depend continuously on θ, then (OPT) is self flattening. For example, a *starlike* set with respect to a point c is one which is "linearly contractible" to c. A corollary of Theorem 4 roughly says if the sets $S(\theta, \gamma_0)$ are continuously varying and starlike, then (OPT) is self flattening. In order to state a precise theorem quickly we restrict attention for the moment to convex sets.

We shall say Γ is *quasi-convex* in w if for each θ and t the sublevel set $S(\theta, t)$ is convex. We also need a mild nondegeneracy condition on the $S(\theta, t)$. Assuming that Γ is continuous, the set $\bigcup_{r<t} S(\theta, r)$ will always be an open subset of $S(\theta, t)$. If it is not the full interior of $S(\theta, t)$, then $\Gamma(e^{i\theta}, \cdot)$ will be constant on some open set. This degenerate behavior causes technical problems and we wish to rule it out. Therefore we say Γ has *degenerate stationary behavior* if $S(\theta, t) - \bigcup_{r<t} S(\theta, r)$ has empty interior.

THEOREM 1. *Suppose Γ is continuous on $\Pi \times \mathbb{C}^N$, is large at ∞, is quasiconvex and does not show degenerate stationary behavior. Let $h_0 \in H^\infty(N)$ solve the problem* (OPT). *Then either* $\gamma_0 = \max_\theta \min_w \Gamma(e^{i\theta}, w)$ *or*

$$\Gamma(e^{i\theta}, h_0(e^{i\theta})) = \gamma_0 \tag{0.2}$$

for almost all θ. Moreover, if the sets $S(\theta, t)$ are uniformly (*in θ*) *strictly convex* (*in t*), *then h_0 is unique; also any strict local optimum is a global optimum.*

Section 1 treats nothing but convex problems and is partially subsumed by Section 2. Section 2 is independent of Section 1 and shows that holomorphic contractibility implies self-flattening. Section 3 is independent of Sections 1 and 2 and shows that either h_0 is badly discontinuous or it makes the objective function flat. Section 4 gives some physical and mathematical examples. It depends only on Section 1.

1. Optima with Convexity Assumptions

Theorem 1 can be proved by means of a Hahn–Banach separation argument. In this section we shall first formulate a result (Theorem 2) which makes the use of separation quite explicit. We then show how Theorem 2 implies Theorem 1. We close the section with a comment concerning the continuity of the optimizing function h_0.

Let $L^\infty(N)$ be the $\mathbb{C}^N$-valued bounded measurable functions on Π. Consider $f \in L^\infty(N)$. Given a point $w \in \mathbb{C}^N$ and θ in $\mathbb{R}$, we say w is in the *essential range of f near $e^{i\theta}$* if for any neighborhoods U of w and V of $e^{i\theta}$, the set $f^{-1}(U) \cap V$ has positive measure. Denote the essential range of f near $e^{i\theta}$ by $\operatorname{essran}(f, \theta)$. Evidently $\operatorname{essran}(f, \theta)$ is closed. On the other hand, if $S_\theta \subseteq \mathbb{C}^N$ is a closed set, then the family

$$\{f \in L^\infty(N)\colon S_\theta \supseteq \operatorname{essran}(f, \theta)\} \tag{1.1}$$

is easily seen to be a closed set. Also if S_θ is convex, then the family (1.1) is also convex.

Let $S \subseteq L^\infty$ be a family of functions. For each θ, define the *local cross section* S_θ of S at θ by

$$S_\theta = \bigcup_{f \in S} \operatorname{essran}(f, \theta). \tag{1.2}$$

We will say S is *local* if

$$\{f \in L^\infty(N)\colon \operatorname{essran}(f, \theta) \subseteq S_\theta \text{ for all } \theta\} = S. \tag{1.3}$$

From the remarks just above, we see that if S is local, and if each S_θ is closed and convex, then S is closed and convex.

Let $C(N)$ denote the space of continuous $\mathbb{C}^N$-valued functions on Π, and $A(N)$ the space of boundary values of holomorphic $\mathbb{C}^N$-valued functions on D which extend continuously to $\bar{D} = D \cup \Pi$.

Theorem 2. *Suppose S is a subset of $L^\infty(N)$ with the following properties:*

(i) *S is local.*

(ii) *The local cross sections S_θ are closed, convex, and bounded independent of θ.*

(iii) *S has nonempty interior.*

(iv) *Every element of S is a pointwise limit of functions in $S \cap C(N)$.*

(v) *The interior of S is disjoint from $A(N)$.*

Then any function $h_0 \in S \cap H^\infty(N)$ has values $h_0(e^{i\theta})$ which lie in the boundary ∂S_θ of S_θ for almost all θ. Moreover, if the S_θ are uniformly strictly convex, then h_0 is unique.

Proof. As was remarked above, properties (i) and (ii) of S imply that S is convex. Since S has nonempty interior, the Hahn–Banach Theorem plus property (v) of S implies there is a nonzero linear functional λ on $C(N)$ such that

$$\operatorname{Re} \lambda(S \cap C(N)) \leqslant a \leqslant \operatorname{Re} \lambda(A(N)).$$

Since $A(N)$ is a subspace of $C(N)$, the values $\operatorname{Re} \lambda(A(N))$ must be either all of $\mathbb{R}$ or 0. Clearly the inequality implies $\operatorname{Re} \lambda(A(N)) = 0$. Hence the Riesz Representation Theorem and the theorem of F. and M. Riesz imply that λ may be represented by an element $l \in H_0^1(N)$, the space of N-tuples of functions in H_0^1, the subspace of the Hardy class H^1 on Π whose elements have vanishing 0th Fourier coefficient. That is,

$$\lambda(f) = \int_\Pi l \cdot f \, d\theta, \qquad f \in C(N), \tag{1.4}$$

where $l = (l_1, \ldots, l_N)$, with $l_i \in H_0^1$, and $f = (f_1, \ldots, f_N)$ with $f_i \in C$, and $l \cdot f = \sum l_i f_i$. The right-hand side of Eq. (1.4) clearly makes sense for all $f \in L^\infty(N)$, and we use Eq. (1.4) to extend λ to all of $L^\infty(N)$.

By assumption (iv), for any $f \in S$ we can find a sequence $f_n \in S \cap C(N)$ which converges pointwise to f. Since S is bounded, the Lebesgue Dominated Convergence Theorem applies and implies that

$$0 \leqslant \operatorname{Re} \int_\Pi l \cdot f \, d\theta.$$

Hence $\operatorname{Re} \lambda(f) \leqslant 0$ for all $f \in S$.

Now consider $h_0 \in S \cap H^\infty(N)$, and suppose that for θ in some set α of positive measure in Π the function h_0 takes values at distance $\varepsilon > 0$ from ∂S_θ. (Here we measure distance by means of any convenient norm η on $\mathbb{C}^N$.) Let f be any function in $L^\infty(N)$ such that f is supported on α and $\|\eta(f)\|_\infty < \varepsilon$. Since S is local the sum $h_0 + f$ will still be in S, whence $0 \leqslant \operatorname{Re} \lambda(h_0 + f) = \operatorname{Re} \lambda(f)$. We conclude that the representing function l for λ must vanish on α. But this is impossible for $l \in H^1(N)$. This contradiction shows that necessarily $h_0(\theta) \in \partial S_\theta$ almost everywhere. This proves Theorem 2 except for uniqueness.

To prove uniqueness suppose that an h_0 and h_0^1 both exist. Then $K \triangleq$

$(h_0 + h_0^1)/2$ satisfies the hypothesis of the theorem, thus hugs the boundary of S, contradicting strict convexity.

It is a fairly simple matter to derive Theorem 1 from Theorem 2.

Proof of Theorem 1. Let Γ be the objective function of Theorem 1, and let γ_0 be the best value for the problem (OPT). We assume that $\gamma_0 > \max_\theta \min_w \Gamma(e^{i\theta}, w)$. Then for each θ the sublevel set $S(\theta, \gamma_0)$ of definition (0.1) is convex and has nonempty interior. Define

$$S = \{f \in L^\infty(N) \colon \text{essran}(f, \theta) \subseteq S(\theta, \gamma_0)\}. \tag{1.5}$$

We claim that S satisfies conditions (i)–(iv) of Theorem 2. Furthermore the local cross section S_θ of S is just $S_\theta = S(\theta, \gamma_0)$.

Assuming our claim is true, we see Theorem 1 follows. Indeed, the definition of γ_0, together with the assumption that Γ does not have degenerate stationarities implies condition (v) of Theorem 2 is also satisfied. Further, an optimum solution h_0 as in Theorem 1 would be in $S \cap H^\infty(N)$. The conclusion of Theorem 2, together with the fact that $S_\theta = S(\theta, \gamma_0)$ then implies the assertion (0.2).

It remains to verify our claim. It is obvious that the set S of definition (1.5) is local, for the local cross sections S_θ are clearly contained in the $S(\theta, \gamma_0)$. Hence if $f \in L^\infty(N)$ is such that $\text{essran}(f, \theta) \subseteq S_\theta$ for all θ, then a fortiori $\text{essran}(f, \theta) \subseteq S(\theta, \gamma_0)$, whence $f \in S$. Assumption (ii) will clearly follow from the equality $S_\theta = S(\theta, \gamma_0)$.

Let us write

$$S^0(\theta, \gamma_0) = \bigcup_{\gamma < \gamma_0} S(\theta, \gamma) = \{w \in \mathbb{C}^N \colon \Gamma(e^{i\theta}, w) < \gamma_0\}.$$

Our assumptions on Γ imply that $S^0(\theta, \gamma_0)$ is open and convex, and that $S(\theta, \gamma_0)$ is the closure of $S^0(\theta, \gamma_0)$. Our assumption on γ_0 implies $S^0(\theta, \gamma_0)$ is nonempty for each θ. Moreover, since Γ is continuous, the set

$$\{(e^{i\theta}, w) \colon w \in S^0(\theta, \gamma_0)\}$$

is open in $\Pi \times \mathbb{C}^N$. Standard selection theorems [B-P] allow us to find $f_0 \in C(N)$ such that $f_0(e^{i\theta}) \in S^0(\theta, \gamma_0)$. Then for each θ, the set $S^0(\theta, \gamma_0) - f_0(e^{i\theta}) = U(\theta)$ is an open convex neighborhood of the origin in $\mathbb{C}^N$. Let $p(\theta, w)$ be the Minkowski support function of $U(\theta)$ (see [R-S, Chap. V]). That is,

$$p(\theta, w) = \inf\{t \in \mathbb{R}^+ \colon t^{-1} w \in U(\theta)\}. \tag{1.6}$$

Then for each θ, $p(\theta, w)$ is a continuous, positive homogeneous convex function of w. We claim further that $p(\theta, w)$ is jointly continuous in θ and

w. Let us grant the claim for the moment, and let η be the norm on $\mathbb{C}^N$ used in the proof of Theorem 2. Define a map d from $\Pi \times \mathbb{C}^N$ to itself by the formula

$$d(\theta, w) = \left(\theta, \frac{p(\theta, w - f_0(\theta))}{\eta(w - f_0(\theta))} (w - f_0(\theta)) \right). \tag{1.7}$$

One can check directly that d is a homeomorphism of $\Pi \times \mathbb{C}^N$, and that

$$d(\theta, S(\theta, \gamma_0)) = (\theta, B(\eta, 1)),$$

where

$$B(\eta, 1) = \{ w \in \mathbb{C}^N : \eta(w) \leqslant 1 \}$$

is the unit ball in $\mathbb{C}^N$ with respect to the norm η. Conditions (ii), (iii), and (iv) of Theorem 2, as well as the additional statement about local cross sections of S are completely obvious if $S(\theta, \gamma_0)$ is replaced by $B(\eta, 1)$ for each θ. But these statements may be pulled back from this obvious situation to our given set S by means of the map d defined in formula (1.7). Hence they hold for S also.

Therefore, to complete Theorem 1, it remains only to check that the Minkowski functionals $p(\theta, w)$ defined in (1.6) are continuous in θ and w jointly. Consider a point $w \in \mathbb{C}^N$, and let $t \in \mathbb{R}$ be such that $t^{-1} w \in U(\theta)$. Since the set $\{(\theta, u) : u \in U(\theta)\}$ is open in $\Pi \times \mathbb{C}^N$, we see that $t^{-1} w \in U(\theta')$ for θ' sufficiently close to θ. It follows from the definition (1.6) of $p(\theta, w)$ that

$$\lim_{\theta' \to \theta} p(\theta', w) \leqslant p(\theta, w), \qquad w \in \mathbb{C}^N.$$

Suppose for some $w \in \mathbb{C}^N$ and some θ, one had

$$\liminf_{\theta' \to \theta} p(\theta', w) \leqslant (1 - \delta)\, p(\theta, w)$$

for some $\delta > 0$. We may scale w so that

$$1 < p(\theta, w) < (1 + \delta). \tag{1.8}$$

Then we can find θ' converging to θ such that $p(\theta', w) \leqslant (1 - \delta)$. It follows that $f_0(\theta') + w \in S(\theta', \gamma_0)$, or equivalently $\Gamma(e^{i\theta'}, f_0(\theta') + w) \leqslant \gamma_0$. Letting θ' approach θ, we conclude that $\Gamma(e^{i\theta}, f_0(\theta) + w) \leqslant \gamma_0$, whence $f_0(\theta) + w \in S(\theta, \gamma_0)$. But inequality (1.8) implies that $f_0(\theta) + w$ is not in the closure of $S^0(\theta, \gamma_0)$, contradicting our assumption that Γ has no degenerate stationary behavior. Review of the above argument shows that, rather than keep w fixed, we could have chosen for each θ' a $w' = w'(\theta')$, such that

$w' \to w$ as $\theta' \to \theta$. Therefore we have established the continuity of $p(\theta, w)$, and Theorem 1 is proved.

The optimizing function h_0 for (OPT) of course cannot be expected to be continuous in general. However we can show that one can come arbitrarily close to the optimum value γ_0 for (OPT) by means of continuous functions.

THEOREM 3. *Let Γ be as in Theorem* 1, *and let h_0 be an optimizing function for the problem* (OPT). *For $\delta \geqslant 0$, set*

$$h_\delta(e^{i\theta}) = h_0((1-\delta)\, e^{i\theta}).$$

That is, h_δ is the restriction to the circle $|z| = 1-\delta$ of the holomorphic extension of h_0 into D. Then

$$\lim_{\delta \to 0} \sup_{\theta \in \mathbb{R}} \Gamma(e^{i\theta}, h_\delta(e^{i\theta})) = \gamma_0. \tag{1.9}$$

Proof. For a given θ and $\varepsilon > 0$, consider the sublevel set $S^0(\theta, \gamma_0 + \varepsilon)$. This is an open convex set containing a neighborhood of $S(\theta, \gamma_0)$. By the continuity in θ of the $S(\theta, \gamma_0)$, demonstrated above in the last part of the proof of Theorem 1, we know that $S(\psi, \gamma_0) \subseteq S^0(\theta, \gamma_0 + \varepsilon)$ for ψ sufficiently near θ, say for $|\theta - \psi| < \mu$. We have

$$h_\delta(e^{i\theta}) = \int_\Pi K((1-\delta)\, e^{i\theta}, e^{i\psi})\, h_0(e^{i\psi})\, d\psi,$$

where K is the Poisson kernel, which satisfies

$$K \geqslant 0, \quad \int_\Pi K((1-\delta)\, e^{i\theta}, e^{i\psi})\, d\psi = 1.$$

Hence $h_\delta(e^{i\theta})$ is in the convex hull of the values of h_0. Further, as $\delta \to 0$, the mass of $K((1-\delta)\, e^{i\theta}, e^{i\psi})$ becomes concentrated near $\psi = \theta$. Thus we can write

$$h_\delta(e^{i\theta}) = a_\delta(e^{i\theta}) + b_\delta(e^{i\theta}),$$

where

(i) $a_\delta(e^{i\theta})$ is a convex combination of the $h_0(e^{i\psi})$ for $|\psi - \theta| < \mu$;

(ii) $b_\delta \to 0$ uniformly as $\delta \to 0$.

Therefore, since $h_0(e^{i\psi}) \in S(\psi, \gamma_0) \subseteq S(\theta, \gamma_0 + \varepsilon)$ for $|\psi - \theta| < \mu$, we have $a_\delta(e^{i\theta}) \in S(\theta, \gamma_0 + \varepsilon)$. Then since $b_\delta \to 0$, for δ sufficiently small we will have $h_\delta(e^{i\theta}) \in S(\theta, \gamma_0 + 2\varepsilon)$ since Π is compact. In other words $\Gamma(e^{i\theta}, h_\delta(e^{i\theta})) < \gamma_0 + 2\varepsilon$ for all θ and δ small. Since ε is arbitrary, Theorem 3 follows.

Remark a. An optimist might hope that Theorem 1 might be improved to say that the optimizing h_0 could be taken to have values in the extreme points of the sets $S(\theta, \gamma_0)$. This is false as the following example shows. Pick a continuous $\mathbb{C}$-valued function f on Π. Define Γ on $\Pi \times \mathbb{C}$ by the recipe

$$\Gamma(\theta, w) = \max\{|\mathrm{Re}(w - f(e^{i\theta})|, |\mathrm{Im}(w - f(e^{i\theta})|\}.$$

The sublevel sets $S(\theta, t)$ for this Γ are just squares of side $2t$ centered at $f(e^{i\theta})$. The extreme points of $S(\theta, t)$ are the corners $f(e^{i\theta}) + t(\pm 1 \pm i)$ of the square. Thus if h_0 is an optimizing function for (OPT) with this Γ, and if h_0 takes values in the extreme points, then on some set of positive measure we have

$$h_0(e^{i\theta}) = f(e^{i\theta}) + c,$$

where c is constant. If f is a rational function, then by analytic continuation, we have $h_0 = f + c$ identically; but if f has a pole inside the disk D, this is impossible.

Remark b. The first alternative in Theorem 1, namely

$$\gamma_0 = \max_{\theta} \min_{w} \Gamma(e^{i\theta}, w)$$

can indeed occur. For example, let b be a nonconstant positive real-valued function on Π, and define Γ on $\Pi \times \mathbb{C}$ by

$$\Gamma(e^{i\theta}, w) = b(e^{i\theta}) + |w|^2,$$

where $|\cdot|$ is the usual absolute value on $\mathbb{C}$. Clearly $\gamma_0 = \|b\|_\infty = \max_\theta \min_w \Gamma(e^{i\theta}, w)$. Also $h_0 = 0$ is an optimizing function for (OPT) with this Γ, and $\Gamma(e^{i\theta}, h_0(e^{i\theta})) = b(e^{i\theta})$ is not constant.

2. Optima with Compressibility

In this section, we relax the condition of convexity on the sublevel sets of Γ, and replace it with a more general and flexible notion, though at a price of additional complication in the formulation and proof of our result.

Let $W \subseteq \mathbb{C}^N$ be a closed set with boundary ∂W. For $w \in W$, and norm η let

$$\delta(w) = \min\{\eta(w - y): y \in \partial W\} \tag{2.1}$$

be the distance to the boundary. We will say W is *holomorphically compressible* if there is a neighborhood U of W and a holomorphic vector field

$$v: \quad U \to \mathbb{C}^N$$

which is directed inward on ∂W, in the sense that there is $\alpha > 0$ so that $w - tv(w)$ is in the interior of W for all $w \in W$ and $0 \leqslant t \leqslant \alpha$. We will say W is *transversally holomorphically compressible* if the vector field v can be chosen so that

$$\delta(w + tv(w)) \geqslant \nu t, \qquad 0 \leqslant t \leqslant \alpha \tag{2.2}$$

for some number $\nu \geqslant 0$. In either of these definitions, we will say w is a *compression field* for W.

If ∂W is a smooth (real) codimension one submanifold of $\mathbb{C}^N$, the condition (2.2) just says that v is transverse to ∂W and pointed inward. We observe that the vector field v is highly nonunique. If p is any holomorphic vector field defined on U such that

$$\sup\{\eta(p(w)) \colon w \in W\} = \mu < \nu$$

then $V' = v + p$ satisfies (2.2) with $\nu' = \nu - \mu$ in place of ν.

We note that all convex sets with nonempty interior are holomorphically compressible. Somewhat more generally, let us say that W is *strictly starlike* with respect to the center y if $w + \lambda(y - w)$ is in the interior of W for $0 < \lambda \leqslant 1$. Then if W is strictly starlike with respect to y, the vector field $v(w) = y - w$ makes W holomorphically compressible. Simply connected bounded domains in $\mathbb{C}$ with smooth boundary are holomorphically compressible. However, annuli, such as $\{z \colon a \leqslant |z| \leqslant b\} \subseteq \mathbb{C}$, for real numbers $a, b > 0$, are not holomorphically compressible.

Now consider a compact set $W \subseteq \Pi \times \mathbb{C}^N$. Let

$$W_\theta = \{w \in W \colon (e^{i\theta}, w) \in W\}, \qquad \theta \in \mathbb{R} \tag{2.2}$$

be the cross section of W above $e^{i\theta} \in \Pi$. We will assume each W_θ is nonempty. We will say the W_θ are a *uniformly transversally holomorphically compressible family* if there is a neighborhood U of $\bigcup_\theta W_\theta$, and a family of vector fields

$$v\colon \quad \Pi \times U \to \mathbb{C}^N$$

such that

(i) $v(e^{i\theta}, w)$ is continuous in w and θ, and holomorphic in w for each fixed θ, and

(ii) relation (2.2) holds for $W = W_\theta$ and $v = v(e^{i\theta}, \cdot)$, with ν and α independent of θ. (2.3)

Remark. The requirement that each $v(e^{i\theta}, \cdot)$ be defined on some U containing all the W_θ make the condition for uniform transversal holomorphic contractibility considerably more stringent than demanding that each W_θ be transversally holomorphically contractible, in some uniform and continuous way. This extra stringency seems undesirable, but the reader will see it is used in the proof of Theorem 4.

Let $S \subseteq L^\infty(N)$ denote a closed bounded set, and let S_θ be its local cross section at θ, as defined in Section 1. We will prove a generalization of Theorem 2 in which compressibility replaces convexity. There would of course be an analogous generalization of Theorem 1; we leave its explicit formulation to the reader.

THEOREM 4. *Let $S \subseteq L^\infty(N)$ be a subset with the following properties:*

(i) *S is local.*

(ii) *The local cross sections S_θ of S are closed, bounded independent of θ, and form a uniformly transversally holomorphically compressible family.*

(iii) *The interior of S is disjoint from $H^\infty(N)$.*

Then any function $h_0 \in S \cap H^\infty(N)$ has values $h_0(e^{i\theta})$ which lie in the boundary ∂S_θ of S_θ for almost all θ.

Actually (iii) can be replaced by the weaker assumption that h_0 is not the sup norm limit of H^∞ functions h_n each contained in the interior of S.

Proof. Let $v(e^{i\theta}, w)$ be a vector field defined on a neighborhood U of $\bigcup_\theta S_\theta$ and such that relation (2.2) holds for each S_θ and $v(e^{i\theta}, \cdot)$. By standard approximation arguments, (Fourier series, partition of unity, etc.) we can approximate v uniformly on $\Pi \times (\bigcup_\theta S_\theta)$ by finite sums of the form

$$\sum_{i=1}^{m} f_i(\theta)\, v_i(w), \tag{2.4}$$

where the $f_i(\theta)$ are continuous $\mathbb{C}$-valued functions, and the $v_i\colon U \to \mathbb{C}^N$ are holomorphic. We have noted above that if v is a compression vector field for a set W and v satisfies (2.2), then any sufficiently close approximation to v also satisfies (2.2), with perhaps a smaller v. Therefore without loss of generality we may assume that our vector field v has the form (2.4).

Consider now $h_0 \in S \cap H^\infty(N)$. For each θ, let δ_θ denote the function δ of definition (2.1) when $W = S_\theta$. Suppose, contrary to Theorem 4, that

$$\delta_\theta(h_0(e^{i\theta})) \geqslant \gamma \tag{2.5}$$

for θ in some set E of Π of measure >0. We wish to modify h_0 to obtain $\tilde{h}_0 \in H^\infty(N)$ satisfying $\delta_\theta(\tilde{h}_0(e^{i\theta})) \geqslant \beta$ for some $\beta > 0$ and all $\theta \in \Pi$. Such an $\tilde{h}_0$ is clearly in the interior of S, a possibility forbidden by the assumptions of Theorem 4. This contradiction would establish the theorem.

To find $\tilde{h}_0$ we appeal to the following lemma which dramatizes the local nature of $H^\infty(N)$, and which seems of independent interest. Indeed, it seems it should be well known, but we lack a convenient reference.

LEMMA 5. *Let $E \subseteq \Pi$ be a set of positive measure. Then given $f \in C(N)$ and $\varepsilon > 0$, there is $h \in H^\infty(N)$ such that*

$$\eta(f(e^{i\theta}) - h(e^{i\theta})) < \varepsilon$$

for all $\theta \in \Pi - E$.

We will assume Lemma 5 for now and finish the proof of Theorem 4. We have at our disposal a compressing vector field $v(e^{i\theta}, w)$ of the form (2.4). We let the set E on which $\delta_\theta(h_0(e^{i\theta})) \geqslant \gamma$ be the set E of Lemma 5, and for each f_i in the summation (2.4), we select an $h_i \in H^\infty$ approximating it so closely that

$$\eta\left(\sum_{i=1}^{m} (f_i(\theta) - h_i(\theta))\, v_i(w)\right) < v/2 \tag{2.6}$$

on $\Pi - E$, where v is as in relation (2.2).

Consider the function

$$\tilde{h}_0(e^{i\theta}) = h_0(e^{i\theta}) + t\left(\sum h_i(\theta)\, v_i(h_0(e^{i\theta}))\right), \qquad 0 \leqslant t \leqslant \alpha. \tag{2.7}$$

On $\Pi - E$, we have, from (2.2), (2.6), and (2.7), the estimate

$$\begin{aligned}\delta_\theta(\tilde{h}_0(e^{i\theta})) &\geqslant \delta_\theta(\tilde{h}_0(e^{i\theta}) + tv(h_0(e^{i\theta})) \\ &\quad - t\eta\left(\sum_{i=1}^{m} (f_i(\theta) - h_i(\theta))\, v_i(h_0(e^{i\theta}))\right) \\ &\geqslant tv/2. \end{aligned} \tag{2.8}$$

On E, we have, from (2.5) and (2.7), the estimate

$$\delta_\theta(\tilde{h}_0(e^{i\theta})) \geqslant \delta_\theta(h_0(e^{i\theta})) - t\eta\left(\sum h_i(\theta)\, v_i(h_0(e^{i\theta}))\right). \tag{2.9}$$

Since the h_i are bounded, it is clear from estimates (2.8) and (2.9) that for small positive t, $\delta_0(\tilde{h}_0(e^{i\theta})) > \beta$ for some small $\beta > 0$, as desired. This concludes Theorem 4.

Proof of Lemma 5. Clearly it is enough to prove the lemma for $N = 1$, because if we can approximate each coordinate f_i of $f \in C(N)$ by $h_i \in H^\infty$, then $h = (h_1, h_2, ..., h_N) \in H^\infty(N)$ approximates f.

Let ξ be the characteristic function of $\Pi - E$. Given $f \in C$ and $h \in H^\infty$, we have $|f - h| \leqslant \varepsilon$ on $\Pi - E$ if and only if

$$|f - h| \leqslant \varepsilon\xi + M(1 - \xi) \tag{2.10}$$

for some suitably large number M. Let $\alpha_{M,\varepsilon}=\alpha$ denote the outer Wiener–Hopf factorization of $(\varepsilon\xi+M(1-\xi))^2$. Then α is in H^∞, and so is α^{-1}, and (2.10) holds if and only if

$$\|\alpha^{-1}f-\alpha^{-1}h\|_\infty\leqslant 1. \tag{2.11}$$

Writing $\alpha^{-1}f=g$ and $\alpha^{-1}h=\kappa$, we see inequality (2.11) is equivalent to

$$\|g-\kappa\|_\infty\leqslant 1 \tag{2.12}$$

for some $\kappa\in H^\infty$. According to a result of Nehari (see [A-A-K]) inequality (2.12) can be achieved if and only if the Hankel operator S_g attached to g has norm bounded by 1.

Recall the definition of the Hankel operator S_g. Let H^2 be the Hardy space of L^2 functions on Π with vanishing negative Fourier coefficients. Let $\bar{H}^2$ be the complex conjugate of H^2, the space of L^2 functions with vanishing positive Fourier coefficients, and let $\bar{H}^2_0$ be the subspace of $\bar{H}^2$ of functions whose 0th Fourier coefficient also vanishes. Then

$$L^2=H^2\oplus\bar{H}_0.$$

Let P be orthogonal projection onto H_2, and $\bar{P}=1-P$, orthogonal projection onto $\bar{H}^2_0$. Then the Hankel operator is a map

$$S_g\colon H^2\to\bar{H}^2_0$$

defined by

$$H_g(h)=\bar{P}(gh),\qquad h\in H^2.$$

For $g=\alpha^{-1}f$, with $\alpha^{-1}\in H^\infty$, straightforward manipulation yields that

$$\begin{aligned}H_g(h)&=H_{\alpha^{-1}f}(h)=\bar{P}\alpha^{-1}(\bar{P}+P)\,f(h)\\&=\bar{P}\alpha^{-1}\bar{P}(fh)\\&=(\bar{P}\alpha^{-1}\bar{P})\,H_f(h).\end{aligned}$$

Since f is continuous, another result of Nehari [P] says H_f is compact. Recall that $\alpha=\alpha_{M,\varepsilon}$. We are interested in choosing M and ε so that $(\bar{P}\alpha^{-1}\bar{P})\,H_f$ has norm less than 1. Suppose that when $M\to\infty$, the operator $\bar{P}\alpha^{-1}\bar{P}$ converges to zero in the strong operator topology. Then since H_f is compact, the product $(\bar{P}\alpha\bar{P})\,H_f$ will converge to zero in norm, and Lemma 5 will follow.

Consider, therefore, the operator $\bar{P}\alpha^{-1}\bar{P}$. The matrix of this operator with respect to the basis $e^{-in\theta}$, $n\geqslant 1$, of $\bar{H}^2$ is

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \\ 0 & 0 & a_0 & \\ \vdots & & & \end{bmatrix}, \tag{2.13}$$

where the a_i are the Fourier coefficients of α^{-1}. As $M \to \infty$, the function $(\alpha_{M,\varepsilon})^{-1}$ converges to zero uniformly on E. Since E has positive measure, $(\alpha_{M,\varepsilon})^{-1}$ cannot have a nonzero limit in H^2; hence the Fourier coefficients of $(\alpha_{M,\varepsilon})^{-1}$ must converge to zero. From the form (2.13) of $\bar{P}\alpha^{-1}\bar{P}$, we see that if the $a_i \to 0$, then $\bar{P}\alpha^{-1}\bar{P}$ does converge strongly to zero. Thus Lemma 5 is proved.

EXTENSION OF THEOREM 4. *Optimization over $H^\infty(\mathbb{C}^N)$ in Theorem 4 could be replaced by optimization over many other subsets A_1 of $L^\infty(\mathbb{C}^N)$. For example, given $c_0 \in C(\mathbb{C}^N)$ and ϕ rational with unitary values $\phi(e^{i\theta})$ for all θ. Suppose that*

$$A_1 \quad \text{contains the set} \quad c_0 + \phi H^\infty(\mathbb{C}^N)$$

Then Theorem 4 holds with A_1 replacing A.

To prove this note that Lemma 5 holds with $c_0 + \phi H^\infty(\mathbb{C}^N)$ replacing $H^\infty(\mathbb{C}^N)$; then go through the proof of Theorem 4 using this new set.

3. MORE GENERAL Γ

In this section we drop assumptions about the sublevel sets of Γ and investigate the extent to which the extremal properties of minimizing functions for (OPT) persist. Our main result shows that if the objective function is not constant for an optimum h_0, then h_0 must have a severe discontinuity. In this section we take $N = 1$ for simplicity.

The general theorem from which this follows is

THEOREM 6. *Suppose $\Gamma(e^{i\theta}, w)$ is a positive continuously differentiable function and that $h_0 \in H^\infty$ minimizes* (OPT). *Suppose $(\partial\Gamma/\partial w)(e^{i\theta}, h_0(e^{i\theta}))$ is uniformly bounded away from zero. Then either*

(I) *$\Gamma(e^{i\theta}, h_0(e^{i\theta})) \equiv \gamma_0$, for almost all θ, or*

(II) *The L^∞-distance of $\Gamma(e^{i\theta}, h_0(e^{i\theta}))$ to the space $(\partial\Gamma/\partial w)(e^{i\theta}, h_0(e^{i\theta}))[H^\infty + C]$ is γ_0.*

Before giving a proof let us see how the intricate conclusion of the theorem actually gives straightforward results. Abbreviate $\partial\Gamma/\partial w$ to Γ_w and note $\Gamma_{\bar{w}} = \bar{\Gamma}_w$. Suppose h_0 is continuous and the function $\Gamma_w(e^{i\theta}, h_0(e^{i\theta}))$ has inverses in L^∞. Set $g_0(e^{i\theta}) = \Gamma(e^{i\theta}, h_0(e^{i\theta}))$. Then $\text{dist}(g_0, \Gamma_{\bar{w}} C) = \text{dist}(g_0, C) = 0 < \gamma_0$, so (II) does not hold. By the theorem, (I) holds; that is, $g_0 = \gamma_0$ a.e. Thus continuity of h_0 implies $g_0 = \gamma_0$ a.e.

If h_0 is near a continuous function, then g_0 and $\Gamma_w(e^{i\theta}, h_0(e^{i\theta}))$ are near continuous functions so $\text{dist}(g_0, \Gamma_w C)$ is small. Again by the theorem $g_0 = \gamma_0$ a.e. We have just proved the corollary in the Introduction.

Theorem 6 has a peculiar asymmetry; indeed, we would expect that (II) could be replaced by

(II′) *The L^∞-distance of $\Gamma(e^{i\theta}, h_0(e^{i\theta}))$ to the space*

$$\frac{\partial \Gamma}{\partial w}(e^{i\theta}, h_0(e^{i\theta}))[H^\infty + C] + \frac{\partial \Gamma}{\partial \bar{w}}(e^{i\theta}, h_0(e^{i\theta}))[\bar{H}^\infty + C]$$

is γ_0).

The proof of the theorem is based on

LEMMA 7. *Suppose $h_0 \in H^\infty$ is an optimizing function in Theorem 6; set $g_0(e^{i\theta}) = \Gamma(e^{i\theta}, h_0(e^{i\theta}))$ and $a(e^{i\theta}) = \Gamma_w(e^{i\theta}, h_0(e^{i\theta}))$. Assume that $|a|^{-1} \in L^\infty$. Then there is a sequence of functions F_n in L^1 with $aF_n \in H_0^1$ and $\|F_n\|_{L_1} = 1$ such that*

$$\gamma_0 = \lim_{n \to \infty} \int_0^{2\pi} g_0 F_n.$$

Proof. Let M be the submanifold $M = \{\Gamma(e^{i\theta}, h(e^{i\theta})) : h \in H^\infty\}$ of L^∞. By a basic principle of approximation theory [W] any closest point g_0 in M to 0 is also a closest point to 0 from the tangent space $T_{g_0}M$ to M at g_0.

Now the complexification of $T_{g_0}M$ contains $g_0 + \text{clos}\,\tau$, where τ denotes the space

$$\tau = aH^\infty + \bar{a}\bar{H}^\infty.$$

and $\text{clos}\,\tau$ is its norm closure in L^∞. This is because $(d/dt)\,\Gamma(e^{i\theta}, h_0(e^{i\theta}) + th(e^{i\theta}))|_{t_0} = a(e^{i\theta})\,h(e^{i\theta}) + \bar{a}(e^{i\theta})\,h(e^{i\theta})$ is in $T_{g_0}M - g_0$ and substituting ih for h tells us that $i(ah - \bar{a}\bar{h})$ is in $T_{g_0}M - g_0$. Consequently ah and $\bar{a}\bar{h}$ are in the complexification of $T_{g_0}M - g_0$ which thus contains τ. Moreover, the complexification of $T_{g_0}M$ is contained in $g_0 + \text{clos}\,\tau$, because H^∞ is its own tangent space at any point. Since g_0 is real it is a closest point to zero from $T_{g_0}M$ if and only if it is a closest point from the complexification of $T_g M$. Thus we have established that a closest point in $\text{clos}\,\tau$ to g_0 is 0.

At this point we sacrifice a substantial amount of information by using only the fact that the closest point to g_0 from H^∞ is 0. Since L^∞ is the dual space of L^1 a general Hahn–Banach lemma (cf. [G, Chap. IV, Lemma 1.1, (1.1)′]) implies that if $Y \subset L^1$, has $Y^\perp \subset L^\infty$ as its annihilator, then

$$\operatorname{dist}(g_0, Y^\perp) = \sup \left\{ \int_\Pi g_0 f : f \in Y \text{ and } \|f\|_{L^1} = 1 \right\}.$$

By the annihilator $Y^\perp$ of Y we mean

$$Y^\perp = \left\{ g \in L^\infty : \int_\Pi gf = 0 \text{ for all } f \in Y \right\}.$$

The annihilator of H_0^1 is well known [G, Chap. IV] to be H^∞, consequently H^∞ is the annihilator of H_0^1/a. The lemma follows immediately from this.

For perspective on Theorem 6 with (II′) note that we could easily compute the preannihilator of the weak-∗ closure of τ. Unfortunately $\operatorname{clos} \tau$ is typically not weak-∗ closed, so the preannihilator cannot be used to describe $\operatorname{dist}(g_0, \operatorname{clos} \tau)$. Thus what is required to analyze (II′) is an argument which follows exactly the same outline as the one here but which is based on computing the annihilator in $(L^\infty)^*$ of H^∞. Unfortunately $(L^\infty)^*$ is technically difficult to handle.

Proof of Theorem 6. Suppose that $g_0 < \lambda < \gamma_0$ on a set E of positive measure. Since $g_0 \leqslant \gamma_0$, the F_n of the previous lemma must satisfy

$$\int_E |F_n| \to 0$$

or else $|\int_\Pi g_0 F_n| \leqslant \lambda \int_E |F_n| + \gamma_0 \int_{\Pi/E} |F_n| \to \gamma_1 < \gamma_0$. This contradicts the basic property of F_n. If $a^{-1} \in L^\infty$, then $G_n = aF_n$ is in H_0^1. Since $\|G_n\|_1 \leqslant \|a\|_{L^\infty} < \infty$ and $\int_E G_n \to 0$ each Fourier coefficient of G_n converges to zero as $n \to \infty$ (see [G, Chap. V]). A consequence of this is that if p is any trigonometric polynomial

$$\int_\Pi apF_n = \int_\Pi pG_n = \sum_{k=1}^{M} p_{-k} \hat{G}_n(k) \to 0.$$

For any continuous f and $\varepsilon > 0$, there is a trig polynomial p within ε of f; that is $\|f - p\|_\infty < \varepsilon$. So

$$\left| \int_\Pi akF_n \right| = \left| \int_\Pi apF_n + \int_\Pi a(f-p) F_n \right|$$

$$\leqslant \left| \int_\Pi pG_n \right| + \varepsilon \int_\Pi |aF_n|$$

which is $\leqslant 2\varepsilon \|a\|_\infty$ for large enough n. We conclude that

$$\int_\Pi a(H^\infty + C)\, F_n \to 0.$$

Thus we have a sequence $L_n(f) \triangleq \int_\Pi fF_n$ of norm one linear functionals on L^∞, with $\lim_{n\to\infty} L_n(a(H^\infty + C)) = 0$ and $\lim_{n\to\infty} L_n(g_0) = \gamma_0$. Consequently

$$\gamma_0 \leqslant \operatorname{dist}(g_0, a(H^\infty + C))$$

but this is a priori $\leqslant \operatorname{dist}(g_0, aH^\infty) = \gamma_0$. So Theorem 6 is proved.

The method for Theorem 6 also gives

COROLLARY 8. *Suppose* $\Gamma(e^{i\theta}, w) = |Q(e^{i\theta}, w)|^2$ *where* $Q(e^{i\theta}, w)$ *is analytic in* w *for each* θ. *If* $Q(z, w)/Q_w(z, w)$ *is analytic and bounded for* z *in the annulus* $\mathfrak{A} = \{z : r < |z| < 1\}$ *and* w *in a neighborhood of* $h_0(\mathfrak{A})$, *any solution* h_0 *in* (OPT) *satisfies* $|Q(e^{i\theta}, h_0(e^{i\theta}))|^2 = \gamma_0$ *a.e. Furthermore, if* $1/Q_w(e^{i\theta}, h_0(e^{i\theta}))$ *is bounded and analytic in* $\mathfrak{A}$, *then* $Q(e^{i\theta}, h_0(e^{i\theta}))$ *is analytic in* $r < |z| < 1/r$.

Proof. Take a derivative to find that the tangent space τ of Lemma 7 is

$$\tau = Q_w \bar{Q} H^\infty + \bar{Q}_w Q \bar{H}^\infty.$$

Thus we obtain a function G_n in H_0^1 such that

$$\int \frac{|Q|^2}{Q_w \bar{Q}}\, G_n \to \gamma_0.$$

Moreover, if $|g_0|$ is not constant each Fourier coefficient of G_n converges to 0. Set $a(e^{i\theta}) = Q_w(e^{i\theta}, h_0(e^{i\theta}))$ and $g_0(e^{i\theta}) = Q(e^{i\theta}, h_0(e^{i\theta}))$. If g_0/a is in $H^\infty + C$, the integral $\int_\Pi (g_0/a)\, G_n \to 0$ as in the proof of Theorem 7. This contradiction establishes that if $g_0/a \in H^\infty + C$, then $|g_0| = \gamma_0$ a.e. A condition which guarantees that $g_0/a \in H^\infty + C$ is that g_0/a has an analytic continuation from Π to an annulus $r < |z| < 1$. Our hypothesis insures this and so the first assertion of the theorem is proved. Actually a stronger statement is true: for the argument to work g_0/a need not actually belong to $H^\infty + C$, but merely satisfy $\operatorname{dist}(g_0/a, H^\infty + C) < \gamma_0(\lim \|G_n\|_{L^1})^{-1}$. Since $(\gamma_0)^{1/2}(\lim \|G_n\|_{L^1})^{-1} \geqslant \gamma_0(\|a\|_{L^\infty})^{-1}$ we see that $\operatorname{dist}(g_0/a, H^\infty + C) < \gamma_0(\|a\|_{L^\infty})^{-1}$ suffices.

We begin to prove the second assertion of Corollary 9 by noting that a strong statement about g_0 and its relationship to the dual extremal sequence $F_n = G_n/a\bar{g}_0$ is also true. The first observation is that if μ is any invariant mean on L^∞, the linear functional

$$\mu\left(\int_0^{2\pi} fF_n\right)$$

defined for $f \in C$ has by the Baire Riesz and Fejer Riesz theorems [G] the representation

$$\int_0^{2\pi} fF_\infty,$$

where $a\bar{g}_0 F_\infty \triangleq G_\infty$ is in H_0^1. A key property of F_∞ is

$$\int_0^{2\pi} |g_0|^2 F_\infty = \mu\left(\lim_{n\to\infty}\int_0^{2\pi} |g_0|^2 F_n\right) = \gamma_0$$

because it (plus $\gamma_0 = \|g_0\|_{L^\infty}^2$) implies that $\int_0^{2\pi} |F_\infty| = 1$. This throws us into the equality case of Hölder's inequality, so

$$A \triangleq \frac{g_0}{a} G_\infty = \bar{g}_0 g_0 F_\infty = |F_\infty|. \tag{3.1}$$

Since $(g_0/a)\, G_\infty$ is analytic and bounded in $r < |z| < 1$ and is real on Π, Schwartz' reflection implies that A is analytic for $1 < |z| < 1/r$. The immediate problem is analyticity on $|z| = 1$. Since any ρ between r and $1/r$ produces an integrable function $A(\rho e^{i\theta})$ of θ, Morera's theorem can be used to obtain that A equals a function which is analytic on all of the annulus $r < |z| < 1/r$. To obtain analyticity of g_0 note that $|g_0| \equiv$ constant on Π implies that it reflects to a "pseudomeromorphic" function on the annulus. Intuitively any zero such a function has on Π must be very bad—so bad that $G_\infty/a = A/g_0$ has a very bad singularity on Π. However, the second hypothesis of Corollary 8 implies $1/a$ is bounded analytic on $r < |z| < 1$ and since $G_\infty \in H_0^1$ the singularities of G_∞/a cannot be bad. This contradicts g_0 having zeros and consequently singularities on Π. The rigorous estimates of singularity strength required in this argument are in Lemma 4.5 [Ru].

COROLLARY 10. *If Q in Corollary 8 also is analytic for z in $r < |z| < 1/r$ and w is in $h_0(\{s: r < |z| < 1/r\})$ and there is a $\delta > 0$ such that $|(\partial Q/\partial w)(z, w)| > \delta$ for all z, w in this region, then h_0 is analytic in a neighborhood of Π. Moreover, the winding number of $Q(e^{i\theta}, h_0(e^{i\theta}))/Q_w(e^{i\theta}, h_0(e^{i\theta}))$ is negative.*

Proof. Let M be the function for which

$$M(z, Q(z, w)) = w.$$

That is, $M(z, \ )$ for each z is the inverse of $Q(z, \ )$. It exists for $r < |z| < 1/r$ and $r < |w| < 1/r$ and is an analytic function of both variables, since Q_w does not vanish (see the implicit function theorem in [L]). By Hartog's theorem M is jointly analytic, and Corollary 8 implies analyticity of g_0, thus

$$h_0(z) = M(z, g_0(z))$$

is analytic for z near $|z| = 1$.

To determine the winding number of g_0/a we use Exercise 4 of Chap. 4 [G]. Since g_0/a is continuous and 0 is the closest point to it from H^∞, this exercise in [G] says that g_0/a has negative winding number about the origin.

Remark. At first glance the hypothesis of Corollary 8 seems discouragingly strong. However, if we shift to the viewpoint of Section 2 we see that Corollary 8 fits in very well. The hypothesis of Corollary 8 guarantees that the function $v(e^{i\theta}, w) = Q(e^{i\theta}, w)/Q_w(e^{i\theta}, w)$ is a vector field analytic for $w \in h_0(\mathfrak{A})$ and at each $w \varepsilon$ boundary of $S_\theta(\gamma_0)$ it is directed orthogonally to boundary $S_\theta(\gamma_0)$. Thus $-v$ satisfies the holomorphic compressibility of Section 2.

Extensions of Theorem 6. (A) *Optimization over H^∞ in Theorem* 6 *could be replaced by optimization over many other subsets N of L^∞. For example, suppose $N \subset L^\infty$ has tangent cone $T_{h_0}N$ at the optimum h_0 which contains an affine space of functions of the form $h_0 + kH^\infty$ for some L^∞ function k whose inverse is L^∞ of positive measure. Then Theorem* 6 *holds as is—except the space in* (II) *is*

$$T_{g_0}N + k\Gamma_w(e^{i\theta}, h_0(e^{i\theta}))(H^\infty + C).$$

To prove this we begin with the chain rule which implies $T_{g_0}N \supset g_0 + kaH^\infty + \bar{k}\bar{a}\bar{H}^\infty$. The remainder of the argument is unchanged.

(B) *Π could be replaced by ∂D for any simply connected domain D having rectifiable boundary.*

4. Applications of Theorems 1 and 2

Classical

These are very much in the spirit of the work of Steven Fisher [F1, F2].

(1) Let D be a domain with rectifiable boundary ∂D. Let p be a point in D. Define

$$A \text{ at } A_1 = \{f \in A : f(p) = 0, f'(p) = 1\}.$$

Let $\Gamma(w) = w$. We know that the optimum h_0 is a multiple of the Riemann map from D to the disk. Theorem 1 (extended from Π to ∂D and to A_1) simply says the Riemann map takes ∂D to Π almost everywhere.

(2) Let z_j, w_j for $j = 1,..., L$ be complex numbers. Let

$$A_1 = \{f \in A: f(z_j) = w_j \text{ for } j = 1,..., L\}$$

and $\Gamma(w) = w$. Then (OPT) is just the Nevanlinna–Pick interpolation problem. Theorem I (extended) says the optimum interpolating function has constant modulus a.e. Very general interpolating problems with higher derivative conditions, and matrix valued functions still succumb to the theorem.

Engineering

We begin with the general type of application we had in mind. Every two terminal linear time invariant causal circuit corresponds to a function h in H^∞ of the right half plane (R.H.P.). The imaginary axis parameterizes frequency of operation of the circuit, so $h(\omega)$ describes the behavior of the circuit at frequency ω. Now various quantities of interest are simply functions $\Gamma(i\omega, h(i\omega))$ of $h(i\omega)$. Frequently one wants to minimize such a quantity over $h \in H^\infty$ R.H.P. In a worst case analysis, one is most fearful of the frequency ω_0 at which

$$\sup_\omega \Gamma(i\omega, h(i\omega))$$

is achieved and this is what one wants to minimize over H^∞. Clearly, this is equivalent to the OPT problem ($M = N = 1$) on page one of this paper. Frequently one has additional constraints on h of the form

$$\Gamma_j(i\omega, h(i\omega)) \leqslant 1 \qquad \text{for} \quad j = 2,..., M.$$

Such an optimization problem is equivalent to OPT for $N = 1$ and arbitrary M. Finally, if our circuit has more than two terminals it corresponds to a matrix valued function h. This gives rise to OPT with $N > 1$.

Many physical functions Γ have sublevel sets which are disks. The simple general reason why this is true is that many functions which arise are linear fractional because series and parallel connection of circuits are both linear fractional operations; to wit, if the impedance of two circuits is z_1 and z_2, respectively, then connecting them in series (resp. parallel) gives a circuit whose impedance is $z = z_1 + z_2$ (resp. $1/z = (1/z_1) + (1/z_2)$). The sublevel sets of any linear fractional map are disks. Our problem OPT in such circumstances can be solved explicitly [H1, H2, H3]. When $N > 1$ disks are far less common as sublevel sets.

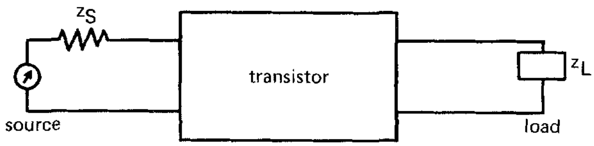

FIGURE 1

The original motivation for this paper was the case where each $S(\theta, t)$ is the intersection of disks. Such problems arise in optimization subject to a constraint; $M > 1$ and $N = 1$. Certainly the intersection of disks is convex so Theorems 1 and 2 apply. We conclude with a specific example, see [H4; H5, Sect. 3A].

We are given a (unilateral) transistor whose scattering function is $S = \begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}$. It typically is employed in an amplifier (Fig. 1), where z_S and z_L are the source and load impedance functions. Suppose z_L is fixed and we want to select z_S to make the power gain

$$G_{T_u} = |S_{21}|^2 \frac{(1 - |\Gamma_S|^2)}{|1 - S_{11}\Gamma_S|^2} \cdot \frac{(1 - |\Gamma_L|^2)}{|1 - S_{22}\Gamma_L|^2} \tag{4.1}$$

large and the noise figure

$$F = F_{\min} + 4r_n \frac{|\Gamma_S - \Gamma_0|^2}{(1 - |\Gamma_S|^2)\,|1 + \Gamma_0|^2} \tag{4.2}$$

small. Here $\Gamma_S = (z_S - 1)/(z_S + 1)$ and $\Gamma_L = (z_L - 1)/(z_L + 1)$ are the reflection coëfficients of source and load. The parameters $F_{\min}$, Γ_0, r_n are noise parameters which are given with the transistor.

The only variable in (4.1) and (4.2) is Γ_S; all other functions are fixed. Thus they have the form

$$G_{T_u}(i\omega) = \Gamma_1(i\omega, \Gamma_S(i\omega)), \qquad F(i\omega) = \Gamma_2(i\omega, \Gamma_S(i\omega)).$$

One observes for fixed ω that the level curves of Γ_1 are circles; also for Γ_2. This obviously helps only at one fixed frequency. The problem of optimizing over all frequencies has not been solved explicitly. However, since the sublevel set

$$S_\omega = \{w \colon \Gamma_1(i\omega, w) < c_1 \text{ and } \Gamma_2(i\omega, w) < c_2\}$$

is convex Theorem 2 applies to give a qualitative result provided that all S_ω are nonempty. Now S_ω nonempty is equivalent to $c_1 < |S_{21}|^2(1 - |S_{11}|^2)^{-1} (1 - |S_{22}|^2)^{-1} \triangleq \max$ and $c_2 > F_{\min}$ by the definition of Γ_1 and Γ_2.

THEOREM 11. *Suppose that Γ_S is chosen optimally but that the gain it produces is at no frequency max and $F(i\omega)$ never equals $F_{\min}$. Then the frequency axis breaks into two sets E_G and E_N such that on E_G the gain $G_{T_u}(i\omega)$ of the amplifier is constant and on E_N the noise figure $F(i\omega)$ is constant.*

We should mention that the optimal Γ_S may not be realizable by a physical circuit and may just be a limit (wk^*) of functions Γ_S^n which are.

Finally, lest ye be deceived by convexity, we remark that there are system examples which produce nonconvex regions (such as lunes).

Note added in proof. A paper by Helton shows under the hypothesis of corollary (Sect. 3) that f_0 is a local optimum if and only if $\Gamma(e^{i\theta}, h_0(e^{i\theta})) \equiv \gamma_0$ and the winding number of $(\partial\Gamma/\partial z)(e^{i\theta}, h_0(e^{i\theta}))$ about zero is positive. One by Helton, Schwartz, and Warschawski shows that when $N=1$ are all smooth "nondegenerate" Γ have the self flattening property.

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